POLLARD WAVES WITH UNDERLYING CURRENTS

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Abstract. We construct an exact solution of the geophysical fluid dynamics governing equations which models waves in the presence of underlying currents, extending Pollard's wave solution to admit underlying meridional and vertical currents at mid-latitudes. We show that these currents cannot be prescribed arbitrarily, but rather must be constant and form a velocity vector which is parallel to the Earth's rotation vector. At the equator, there is freedom to prescribe a meridional current term which may vary both zonally and vertically.

1. Introduction

In this paper we construct an exact solution for the geophysical fluid dynamics (GFD) governing equations in the $f-$plane approximation. This nonlinear three-dimensional solution is explicitly prescribed in terms of Lagrangian coordinates in a rotating system, and extends the celebrated Pollard's wave solution in the sense that it describes zonally-propagating water waves in the presence of meridional and vertical currents. The construction of exact solutions in fluid mechanics is an intricate and unyielding process in general, as evidenced by the scarcity of such solutions. Exact solutions describing water waves are extremely rare, and Gerstner's wave [15] is the only known solution of the nonlinear gravity wave problem with non-trivial free-surface. This solution is two-dimensional, nonlinear and pertains to non-rotating systems. The extension of Gerstner-type solutions to accommodate a rotating system, thereby incorporating Coriolis forces, was achieved by Pollard [38] with his construction of a nonlinear three-dimensional solution to the GFD governing equations in the $f-$plane approximation, which applies at fixed, but arbitrary latitudes, with the exception of the Poles.

The Pollard surface wave is not just a mathematically interesting generalisation of Gerstner's solution, but in the geophysical context it accords with Ursell's observation [40] that Coriolis forces prohibit the existence of large scale Stokes waves with accompanying mass

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transport, implying that such large scale waves should correspond to Gerstner-type waves. Ursell also posited that fluid particle motion should occur in a (locally) vertical orbital plane at the equator, while away from the equator the orbital planes have a very slight cross-wave tilt: Pollard's wave possesses these characteristics. From a mathematical perspective, recent focus on Pollard's solution has considered its dynamical validity [39], stability [25], and mass-transport properties [33].

In general, exact solutions offer an invaluable insight into the mathematical structure of a given problem. From a physical perspective, while exact solutions are inherently idealised representations of fluid flows, they provide a foundation upon which more realistic and observable flows may be constructed. Indeed, there has been a spate of recent mathematical activity generating exact Gerstner-like solutions in the equatorial region which model a plethora of geophysical phenomena, cf. [6, 7, 8, 17, 18, 19, 21, 23, 24, 34, 35, 36], and surveys [2, 22, 26]. Furthermore, Gerstner-like solutions obtained in the $f-$ and $\beta-$plane approximations, at an arbitrary latitude and in the presence of an underlying background current, were derived in [3, 4, 14].

The objective of this paper is to consider whether Pollard's solution can be extended in order to incorporate currents in the meridional and vertical directions: in [11], it was shown that Pollard's solution admits a depth-invariant zonal current, resulting in a number of interesting geophysical consequences. In particular, it was shown that the presence of a mean zonal current generates a 'slow mode' wave solution whose period is of similar scale to the inertial period and whose characteristic wavelength is of order 10^4 km.

From a geophysical perspective, it is natural to consider whether meridional and vertical currents can also be incorporated into Pollard's solution. Large-scale meridional currents arise at mid-latitudes when modelling the Ekman transport phenomenon, whereby waves propagate zonally (due to prevailing trade-winds or westerlies) but the underlying fluid transport is in the transverse meridional direction. The convergence (respectively, divergence) of large-scale transported fluid results in Ekman pumping and an associated downwelling (respectively, upwelling) of fluid, which may be modelled as a vertical current, cf. [10, 9, 12, 13, 16, 37, 41]; further mathematical approaches which facilitate strong depth variations in flow solutions, including upwelling/downwelling processes, are given in [27, 28, 29]. In this paper we establish that the structure of Pollard's solution does indeed permit the incorporation of current terms in the meridional and vertical directions, however at mid-latitudes these currents are necessarily constant and constitute a velocity vector which is parallel to the Earth's rotation vector. At the equator, the vertical current vanishes with the Coriolis parameter f, and the solution corresponds to that considered in

[20], whereby the transverse meridional current is no longer necessarily constant, but may vary zonally and vertically.

2. Preliminaries

We consider the nonlinear flow of a fluid described with reference to a rotating Cartesian coordinate system fixed to the surface of the Earth. The x-axis of this reference frame is parallel to the equator, taken positive from west to east, the y-axis points northwards, while the z-axis points vertically upwards, as illustrated in Figure 1.

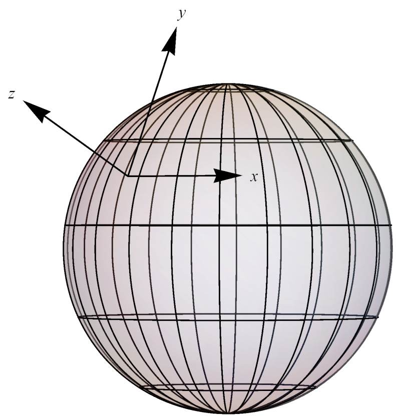

Figure 1. The rotating $\{x, y, z\}$–coordinate system at a fixed point on the surface of the Earth.

Since this coordinate system is not inertial, there is a fictitious force acting on all bodies in this frame of reference, namely the Coriolis force. The GFD equations of motion in this frame of reference are described by the Euler equation [12, 37]

$$u_t + uu_x + vu_y + wu_z + 2\Omega\cos\phi\, w - 2\Omega\sin\phi\, v = -\frac{1}{\rho}P_x$$
$$v_t + uv_x + vv_y + wv_z + 2\Omega\sin\phi\, u = -\frac{1}{\rho}P_y$$
$$w_t + uw_x + vw_y + ww_z - 2\Omega\cos\phi\, u = -\frac{1}{\rho}P_z - g.$$

The constant $\Omega = 7.29\times 10^{-5}\,\mathrm{s}^{-1}$ is the angular velocity of Earth about its axis, with $g = 9.8\,\mathrm{m\,s}^{-2}$ being the acceleration due to gravity on the surface of the Earth, while ϕ denotes the latitude. The functions $u(x, y, z, t), v(x, y, z, t)$ and $w(x, y, z, t)$ denote the velocity of a fluid particle along the x, y and z-axes respectively. Introducing the Coriolis parameters

$$\hat{f} = 2\Omega\cos\phi \quad f = 2\Omega\sin\phi,$$

we obtain the f-plane approximation of the GFD governing equations by assuming that the latitude ϕ (which can be chosen arbitrarily) is fixed, and hence subsequent considerations are valid in a region of restricted latitudinal scope (usually taken to be around a couple of degrees, cf. [12, 16]). Therefore

$$\begin{aligned}
&\frac{Du}{Dt} + \hat{f}w - fv = -\frac{1}{\rho}P_x \\
&\frac{Dv}{Dt} + fu = -\frac{1}{\rho}P_y \\
&\frac{Dw}{Dt} - \hat{f}u = -\frac{1}{\rho}P_z - g,
\end{aligned} \tag{2.1}$$

where

$$\frac{D}{Dt} = \partial_t + \mathbf{u}\cdot\nabla, \quad \mathbf{u} = (u, v, w)$$

is the standard material derivative and the Coriolis parameters f, $\hat{f}$ are now constant. At the equator, where $f = 0$, the system (2.1) is greatly simplified. Additionally, the requirement that the fluid be incompressible is expressed by

$$u_x + v_y + w_z = 0, \tag{2.2}$$

while mass conservation in the flow takes the form

$$\rho_t + u\rho_x + v\rho_y + w\rho_z = 0. \tag{2.3}$$

In the following it shall be assumed that ρ is constant throughout, and so (2.3) holds trivially. The fluid body is infinitely deep and we assume that wave-induced fluid motion vanishes at great-depth.

3. Pollard-like solution

In the current work we consider Pollard wave solutions propagating along the equator in the presence of both transverse, and vertical, underlying currents. This solution of the governing equations (2.1) and (2.2) is prescribed explicitly in terms of Lagrangian labelling variables (q, r, s) as follows:

$$\begin{aligned}
x(q,r,s) &= q - be^{ms}\sin[k(q-ct)], \\
y(q,r,s) &= r - de^{ms}\cos[k(q-ct)] + tV(q,r,s), \\
z(q,r,s) &= s + ae^{ms}\cos[k(q-ct)] + tW(q,r,s).
\end{aligned} \tag{3.1}$$

Here the Lagrangian variables are given by

$$(q, r, s) \in \mathbb{R} \times [-r_0, r_0] \times (-\infty, s_0(r)],$$

where the labelling parameter $r \in [-r_0, r_0]$ is restricted latitudinally to comply with the $f-$plane approximation and, for all $r \in [-r_0, r_0]$, $s_0(r) < 0$ corresponds to the Lagrangian description of the wave surface, cf. Remark 2. System (3.1) features a number of parameters, namely a, b, d, c, k, m, which are precisely determined by the structure of the governing equations (2.1) and (2.2). In the process of demonstrating that relation (3.1) represents a solution to the governing equations we uncover various mathematical relations between these parameters, which are collated in (3.15). Among these parameters, c is the wave-speed, which is prescribed by the dispersion relation (3.16), and k is the wave-number, which is related to the wavelength L by $k = \frac{2\pi}{L}$. We note that $k > 0$, and in order for the wave term in (3.1) to decay with depth it is necessary that $m > 0$. Finally, in the system (3.1), the term V denotes a transverse meridional current, while W represents a vertical current.

Remark 1. Although the vertical scale of ocean currents is much smaller than that of horizontal currents, this is not to say vertical currents are entirely absent from the ocean. One source of meridional and vertical currents in the ocean at mid-latitudes is offered by the Ekman transport and Ekman pumping phenomena [12, 13, 16, 37, 41].

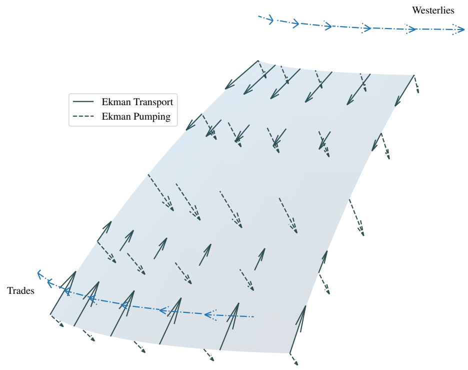

FIGURE 2. Downwelling at mid-latitudes in the Northern Hemisphere.

When the scale of oceanic disturbances is large (and so the Rossby number is low) advective effects become negligible, and the horizontal wind-driven ocean current (U, V) is governed by a balance between the Coriolis force and the surface shear due to the wind. The Westerlies and Trades move in opposite directions, while the Coriolis force causes the surface fluid to move perpendicular to these prevailing winds. It can be shown that in the Northern Hemisphere, where $f > 0$, westerlies induce a mass transport towards the equator ($V < 0$), while trades induce a mass transport towards the pole ($V > 0$). As the westerlies tend to lie north of trades, this Ekman transport generates a convergent meridional flow which induces downwelling of the fluid beneath the surface, which is the Ekman pumping phenomenon. Similarly, Ekman pumping results in fluid upwelling if we reverse the wind directions. In general, downwelling occurs in any anti-cyclonic flow (counter-clockwise in the

Northern Hemisphere) while upwelling occurs in the presence of cyclonic flows (clockwise in the Northern Hemisphere) cf. [12, 16, 37, 41].

A characteristic of Pollard-like waves is they are symmetric in the meridional direction only along the equator, while at mid-latitudes the waves are tilted towards the poles of the respective hemispheres. The angle of tilt is given by $\tan^{-1}\left(\frac{d}{a}\right)$.

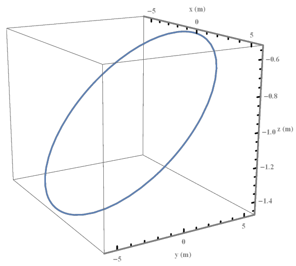

FIGURE 3. Fluid particle trajectory in a frame moving with the current at latitude $\phi = 40°$ N, with parameters $L = 100$m, $a = 1$m, $r = 0$, $s = -1$.

When observed in a reference frame moving with the same velocity $(0, V, W)$ as the underlying current, an individual fluid particle in the flow describes a circle, which can be seen from the following parameterisation of the particle trajectory:

$$\left\{\vec{R}_0 + \delta\cos\theta\vec{\mu} + \delta\sin\theta\vec{\nu}\times\vec{\mu}\right\}_{\theta\in\mathbb{R}}.$$

Here the vectors $\vec{\mu}$ and $\vec{\nu}$ are related to the parameters a, b and d via

$$\vec{\mu} = \left(0, -\frac{d}{b}, \frac{a}{b}\right) \quad \vec{\nu} = \left(0, -\frac{a}{b}, -\frac{d}{b}\right),$$

Pre-empting relation (3.15c), $b^2 = a^2 + d^2 - b^2$, it can be seen that the vector $\vec{R}_0 = (q, r - Vt, s - Wt)$ is the centre of the orbit while $\delta = be^{ms}$ is the circle's radius. In the above expression the parameter θ is arbitrary; for Pollard's wave solution it is defined to be the phase-variable (3.2). This circle is also tilted towards the pole of the corresponding hemisphere at the same angle as the wave-form, as illustrated in Figure 3. In the same reference frame, anticipating once more relations (3.15), the wave profile of a Pollard-type wave corresponds to a trochoid, which is the curve traced by a point fixed at a radial distance be^{ms} from the centre of a disc of radius b, which rolls without slipping.

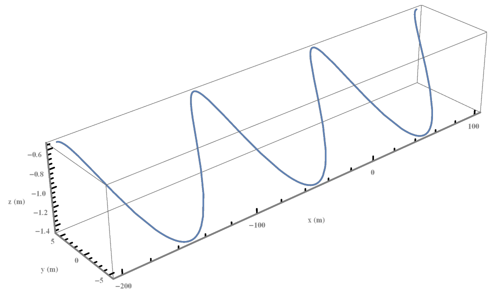

FIGURE 4. A trochoidal Pollard wave at latitude $\phi = 40°$ N, with parameters $L = 100$m, $a = 1$m, $r = 0$, $s = -1$.

3.1. **Admissibility of currents** $V(q, r, s)$ **and** $W(q, r, s)$**.** For notational convenience we introduce the variables

$$\xi = ms \qquad \theta = k(q - ct), \tag{3.2}$$

in which case, the Jacobian matrix of the coordinate transformation (3.1) is now given by

$$\frac{\partial(x, y, z)}{\partial(q, r, s)} = \begin{pmatrix} 1 - bke^{\xi}\cos\theta & V_q t + dke^{\xi}\sin\theta & W_q t - ake^{\xi}\sin\theta \\ 0 & V_r t + 1 & W_r t \\ -bme^{\xi}\sin\theta & V_s t - dme^{\xi}\cos\theta & 1 + W_s t + ame^{\xi}\cos\theta \end{pmatrix}, \tag{3.3}$$

whose determinant is

$$\begin{aligned} J = 1 - abkme^{2\xi} &+ \left[V_r + W_s - (abkmV_r + bdkmW_r)\, e^{2\xi}\right] t \\ + \left[V_r W_s - V_s W_r\right] t^2 &+ \left[bmW_q t + bm(V_r W_q - V_q W_r)t^2\right] e^{\xi}\sin\theta \\ + \left[(am - bk)\,(1 + tV_r) + (dmW_r - bkW_s)t + bk(V_s W_r - V_r W_s)t^2\right] &e^{\xi}\cos\theta. \end{aligned} \tag{3.4}$$

In order to ensure mass conservation, the Jacobian (3.4) should be non-zero and time-independent [1], and so any terms involving t and θ must vanish, resulting in

$$J = 1 - a^2 m^2 e^{2\xi} \neq 0.$$

As a by-product, following computations we obtain from (3.4) the identities:

$$\begin{aligned} &(i)\ am = bk;\ (ii)\ W_r{=}\frac{a}{d}W_s;\ (iii)\ W_q = 0;\ (iv)\ V_r{=}\frac{a}{d}V_s\ (\text{if } W_r, W_s \neq 0); \\ &(v)\ V_q W_r{=}0;\ (vi)\ V_r + W_s = 0. \end{aligned} \tag{3.5}$$

From relation (v) we conclude that either $W_r = 0$, or $V_q = 0$, and both cases are to be considered separately.

3.1.1. $W_r = 0$. If $W_r = 0$, then relation (ii) in (3.5) implies that $W_s = 0$, from which relation (vi) implies $V_r = 0$; we already have $W_q = 0$ in relation (iii). This leads to

$$W_q = W_r = W_s = V_r = 0,$$

and so the vertical current term W is constant. However, we cannot conclude from (iv) that $V_s = 0$, and in order to show that V is also constant we must examine the fluid pressure distribution prescribed by (3.1). The Jacobian (3.3) takes the form

$$J = \begin{pmatrix} 1 - bke^{\xi}\cos\theta & V_q t + dke^{\xi}\sin\theta & -ake^{\xi}\sin\theta \\ 0 & 1 & 0 \\ -bme^{\xi}\sin\theta & V_s t - dme^{\xi}\cos\theta & 1 + ame^{\xi}\cos\theta \end{pmatrix}. \tag{3.6}$$

The Lagrangian formalism used to formulate Gerstner-type solutions is particularly convenient when computing the velocity and acceleration of individual fluid particles along the flow [1, 5], leading to

$$\begin{cases} u = \frac{Dx}{Dt} = bkce^{\xi}\cos(\theta), \\ v = \frac{Dy}{Dt} = V - dkce^{\xi}\sin(\theta), \\ w = \frac{Dz}{Dt} = W + akce^{\xi}\sin(\theta), \end{cases} \tag{3.7}$$

and

$$\begin{cases} \frac{Du}{Dt} = bk^2c^2e^{\xi}\sin(\theta), \\ \frac{Dv}{Dt} = dk^2c^2e^{\xi}\cos(\theta), \\ \frac{Dw}{Dt} = -ak^2c^2e^{\xi}\cos(\theta). \end{cases} \tag{3.8}$$

Substituting (3.7)–(3.8) into the governing equation (2.1) yields a pressure gradient in (x, y, z)-coordinates throughout the flow given by

$$\begin{aligned} P_x &= \rho\left(fV - \hat{f}W\right) - \rho kc\left(bkc + \hat{f}a + fd\right)e^{\xi}\sin\theta, \\ P_y &= -\rho kc\,(dkc + fb)\,e^{\xi}\cos\theta, \\ P_z &= -\rho g + \rho kc\left(akc + \hat{f}b\right)e^{\xi}\cos\theta. \end{aligned} \tag{3.9}$$

For simplicity, we set $\rho = 1$ in what follows. Multiplying (3.9) by the Jacobian matrix (3.6) we deduce that the pressure gradient with respect to labelling variables is

$$
\begin{aligned}
P_q &= fV - \hat{f}W - k\left[b\left(fV - \hat{f}W\right) - c(dkc + bf)V_q t\right] e^{\xi}\cos\theta \\
&\quad + k\left[ga - c(bkc + a\hat{f} + df)\right] e^{\xi}\sin\theta + c^2k^3\left[b^2 - d^2 - a^2\right] e^{2\xi}\sin\theta\cos\theta, \\
P_r &= -kc(dkc + bf)e^{\xi}\cos\theta, \\
P_s &= -g - bm\left[fV - \hat{f}W\right] e^{\xi}\sin\theta + bckm\left[bck + a\hat{f} + df\right] e^{2\xi} \\
&\quad + \left[ac^2k^2 + bck\hat{f} - gam - kc(dkc + fb)V_s t\right] e^{\xi}\cos\theta \\
&\quad + c^2k^2m\left[a^2 + d^2 - b^2\right] e^{2\xi}\cos^2\theta.
\end{aligned}
$$

Applying the compatibility condition $P_{rq} = P_{qr} = 0$, we have

$$dkc + bf = 0. \tag{3.10}$$

The compatibility conditions $P_{qs} = P_{sq}$ require $V_s = 0$ away from the equator, where $f = 0$. Additionally, we have

$$ac^2k^3 = bc^2k^2m + m\left(cdk - bV_q\right) f, \tag{3.11}$$

however due to (3.10) we cannot yet conclude that $V_q = 0$. Integrating the pressure with respect to s, we get

$$
\begin{aligned}
P &= P_0(q,r) - gs - b\left[fV - \hat{f}W\right] e^{\xi}\sin\theta + \frac{bck}{2}\left[bck + a\hat{f} + df\right] e^{2\xi} \\
&\quad + \left[\frac{ac^2k^2}{m} + \frac{bck}{m}\hat{f} - ga\right] e^{\xi}\cos\theta + \frac{c^2k^2}{2}\left[a^2 + d^2 - b^2\right] e^{2\xi}\cos^2\theta.
\end{aligned} \tag{3.12}
$$

Remark 2. In the absence of underlying currents, the free-surface $z = \eta(x,y,t)$ for Pollard's wave is prescribed by setting $s = s_0(r)$ where, for fixed r, the value $s_0(r)$ represents the solution of $P(q,r,s_0(r)) = P_{atm}$ in (3.12) where P_{atm} is the constant atmospheric pressure. A unique solution exists by the implicit function theorem. This approach enforces time independence in the pressure distribution (3.12). The dynamic and kinematic boundary conditions on the free surface are given in Eulerian coordinates by

$$
\begin{cases}
P = P_{atm} \\
w = \eta_t + u\eta_x + v\eta_y
\end{cases}
\qquad \text{on } z = \eta(x,y,t),
$$

and these are automatically satisfied by this prescription. This prescription of a free-surface still works mathematically in the general setting of currents being considered here however, from a physical perspective, away from the Equatorial region this solution would not offer a realistic representation of fluid flow in the near-surface layer.

In light of the above remark, we assume time independence in the pressure distribution, which leads to the relations:

$$\begin{aligned} fV &= \hat{f}W, \\ ac^2k^2 + bck\hat{f} &= gam \\ a^2 + d^2 &= b^2. \end{aligned} \tag{3.13}$$

It follows from the first relation in (3.13) that, away from the Equatorial region ($f \neq 0$), we must have V is constant, and so $V_q = 0$, since W is constant. The condition (3.11) above becomes

$$ac^2k^2 = bc^2km + cdmf.$$

The pressure distribution defined by the Pollard-like solution (3.1) is then prescribed by

$$P = P_{atm} - gs + \frac{ga^2m}{2}e^{2ms},$$

where P_{atm} is the constant atmospheric pressure.

3.1.2. $V_q = 0$. If $V_q = 0$ then the Jacobian takes the form

$$J = \begin{pmatrix} 1 - bke^{\xi}\cos\theta & dke^{\xi}\sin\theta & -ake^{\xi}\sin\theta \\ 0 & V_rt + 1 & W_rt \\ -bme^{\xi}\sin\theta & V_st - dme^{\xi}\cos\theta & 1 + W_st + ame^{\xi}\cos\theta \end{pmatrix}, \tag{3.14}$$

and multiplying (3.9) by the Jacobian matrix (3.14) gives

$$\begin{aligned} P_q &= fV - \hat{f}W + \left[gak - kc(bkc + \hat{f}a + fd)\right] e^{\xi}\sin\theta \\ &\quad - bk\left[fV - \hat{f}W\right] e^{\xi}\cos\theta \\ &\quad + k^3c^2\left[b^2 - a^2 - d^2\right] e^{2\xi}\sin\theta\cos\theta, \\ P_r &= \left[kc(akc + \hat{f}b)W_rt - kc(dkc + fb)V_rt - kc(dkc + fb)\right] e^{\xi}\cos\theta - gW_rt, \\ P_s &= -g - gW_st - bm\left[fV - \hat{f}W\right] e^{\xi}\sin\theta + bckm(bck + a\hat{f} + df)e^{2\xi} \\ &\quad + \left[kc(akc + \hat{f}b) - kc(dkc + fb)V_st + kc(akc + \hat{f}b)W_st - gam\right] e^{\xi}\cos\theta \\ &\quad + \left[c^2k^2m\left(a^2 + d^2 - b^2\right)\right] e^{2\xi}\cos^2\theta. \end{aligned}$$

Since mixed derivatives are equal we have $P_{qr} = P_{rq}$, which leads to

$$\begin{aligned} &fV_r - \hat{f}W_r - bk(fV_r - \hat{f}W_r)e^{\xi}\cos\theta \\ &= -k\left[kc(akc + \hat{f}b)W_rt - kc(dkc + fb)V_rt - kc(dkc + fb)\right] e^{\xi}\sin\theta, \end{aligned}$$

and so we infer that

$$cdk + bf = 0, \qquad fV_r = \hat{f}W_r, \qquad (akc + \hat{f}b)W_r = 0.$$

From direction computation we can see that the equality $P_{rs} = P_{sr}$ provides no additional relations, while the equality $P_{sq} = P_{qs}$ leads to

$$fV_s = \hat{f}W_s, \qquad (akc + \hat{f}b)W_s = 0, \qquad a^2 + d^2 = b^2.$$

Implementing these relations, and integrating P_s with respect to s gives

$$P = P_0(q,r) - gs - g\left(W(s) - W(s_0)\right)t - b\left[fV - \hat{f}W\right]e^{\xi}\sin\theta$$
$$+ \frac{bck}{2}(bck + a\hat{f} + df)e^{2\xi} + \left[\frac{kc}{m}(akc + \hat{f}b) - ga\right]e^{\xi}\cos\theta.$$

It is now clear that the assumed time-independence of the pressure distribution compels W to be constant with respect to s, from which (ii) implies that $W_r = 0$, and the remaining conclusions established in section 3.1.1 all follow.

In summary, we have established that the following relations must hold for the Pollard-like wave-current interactions prescribed by (3.1):

$$fV = \hat{f}W; \tag{3.15a}$$
$$am = bk; \tag{3.15b}$$
$$a^2 + d^2 = b^2; \tag{3.15c}$$
$$dkc + bf = 0; \tag{3.15d}$$
$$ac^2k^2 = bc^2km + cdmf; \tag{3.15e}$$
$$ac^2k^2 + bck\hat{f} = gam. \tag{3.15f}$$

Furthermore, throughout the flow we must have

$$1 - a^2m^2e^{2\xi} > 0, \tag{3.15g}$$

and the pressure distribution for the fluid is prescribed by

$$P = P_{atm} - gs + \frac{ga^2m}{2}e^{2ms}. \tag{3.15h}$$

At the equator, the vertical current W vanishes, and the meridional current takes the form $V = V(q,s)$, which corresponds to the purely equatorial solution considered in [20]. Away from the equator, V and W are constant and relation (3.15a), $fV = \hat{f}W$, holds. This relation implies that the vectors $(0, \hat{f}, f)$ and $(0, V, W)$ are parallel. However, $(0, \hat{f}, f)$ is simply the Earth's rotation vector $\Omega\mathbf{k}$ represented in our rotating coordinate system, since

$$\Omega\mathbf{k} = \Omega\cos\phi\mathbf{e}_y + \Omega\sin\phi\mathbf{e}_z = \frac{\hat{f}}{2}\mathbf{e}_y + \frac{f}{2}\mathbf{e}_z,$$

and so the velocity vector $(0, V, W)$ prescribed by the mean-currents is parallel to the Earth's rotation vector.

3.2. **Dispersion relation.** Expressing relations (3.15b) and (3.15d) as

$$b = \frac{am}{k} \quad d = -\frac{fam}{k^2 c},$$

and substituting into the (3.15e) we find

$$m^2(k^2c^2 - f^2) = k^4c^2.$$

Substituting into (3.15f) yields

$$m(k^2c^2 - f^2) = k^2(g - \hat{f}c),$$

which matches the dispersion relation due to Pollard [38]. In particular, we note that the meridional current V and vertical current W play no role in the dispersion relation, we which express as

$$c^2\left(k^2c^2 - f^2\right) = (g - \hat{f}c)^2. \tag{3.16}$$

At the equator, where $f = 0$ and $\hat{f} = 2\Omega$, this dispersion relation simplifies and we deduce

$$c = \frac{-\Omega \pm \sqrt{\Omega^2 + kg}}{k}, \tag{3.17}$$

which agrees with the dispersion relation obtained in [23].

At mid latitude, where $f \neq 0$, the dispersion relation (3.16) may be re-expressed in terms of the non-dimensional variables

$$X = c\sqrt{\frac{k}{g}}, \quad \varepsilon = \frac{f}{\sqrt{gk}}, \quad F = \frac{\hat{f}}{f},$$

and becomes a fourth-order polynomial in X, given by

$$P(X) = X^4 - \varepsilon^2(1 + F^2)X^2 + 2\varepsilon FX - 1 = 0. \tag{3.18}$$

We note that since $P(0) = -1$ and $P(X) \to \infty$ as $X \to \pm\infty$, then the existence of at least one positive and one negative real root is assured.

3.3. **Vorticity.** Inverting the Jacobian matrix (3.3), now subject to the condition that V and W are both constant, we find at once that

$$\frac{\partial\,(q,r,s)}{\partial\,(x,y,z)} = \frac{1}{J}\begin{pmatrix} 1 + bke^{\xi}\cos\theta & -dke^{\xi}\sin\theta & ake^{\xi}\sin\theta \\ 0 & 1 - b^2k^2e^{2\xi} & 0 \\ bme^{\xi}\sin\theta & dme^{\xi}\cos\theta - bdkme^{2\xi} & 1 - bke^{\xi}\cos\theta \end{pmatrix}. \tag{3.19}$$

Since

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{\partial\,(q,r,s)}{\partial\,(x,y,z)}\frac{\partial(u,v,w)}{\partial(q,r,s)},$$

terms which have been computed explicitly in (3.7) and (3.19), we find

$$\begin{pmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{pmatrix} =$$
$$\frac{1}{J}\begin{pmatrix} -bck^2 e^{\xi}\sin\theta & -cdk^2 e^{\xi}(be^{\xi}+\cos\theta) & ack^2 e^{\xi}(bke^{\xi}+\cos\theta) \\ 0 & 0 & 0 \\ bckme^{\xi}(\cos\theta - bke^{\xi}) & -cdkme^{\xi}\sin\theta & bck^2 e^{\xi}\sin\theta \end{pmatrix}.$$

Taking the trace of this matrix, it follows at once that

$$u_x + v_y + w_z = 0,$$

thus confirming the flow is incompressible. Meanwhile, taking the curl of the velocity field yields the vorticity

$$\vec{\omega} = \begin{pmatrix} w_y - v_z \\ u_z - w_x \\ v_x - u_y \end{pmatrix} = \frac{1}{J}\begin{pmatrix} dkmce^{\xi}\sin\theta \\ ck(bm-ak)e^{\xi}\cos\theta - bck^2(ak+bm)e^{2\xi} \\ -bcdk^3 e^{2\xi} - cdk^2 e^{\xi}\cos\theta \end{pmatrix},$$

and so it is clear the vorticity of the flow is fully three-dimensional. This is also known to be the case with Gerstner waves in the presence of a meridional current, cf. [20], however in that framework the vorticity is only three dimensional if the current is not constant. In contrast, the three-dimensional character of the vorticity in this setting follows from the inherent three-dimensional character of the Pollard-type solution itself.

4. Conclusion

We have demonstrated that an extension of Pollard's solution to include meridional and vertical currents is possible, however this involves conditions which are quite restrictive at mid- or high-latitudes from the viewpoint of modelling large scale geophysical fluid motion, particularly due to the presence of the mean vertical current term prescribed by condition (3.15a). Nevertheless, this extended Pollard's flow (3.1) may offer a useful exact solution when modelling fluid motion at the equator, or at low-latitudes when restricted to regions beneath the surface layer. In this regard we note that Gerstner-like internal wave solutions incorporating variable meridional currents in the equatorial region were constructed in [18]. Recently, adaptations of Pollard's surface-wave solution have extended it to describe wave motion on an internal interface in a multi-layered fluid model, cf. [30, 31, 32], and it may be possible in such models to implement a transition layer which curbs the influence of the mean vertical current away from the fluid surface layer.

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